6.4 The Grand-Schmidt Process

Consider again the set of vectors of exercise 3 (Section 6.3).

$$\vec{y} = \begin{bmatrix} -4 \\ 4 \\ 3 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

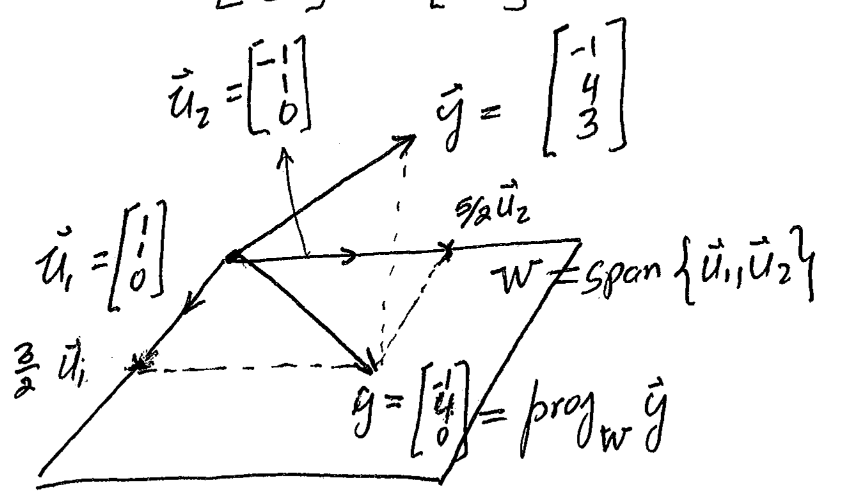
We found that $\vec{u}_1 \cdot \vec{u}_2 = 0$, $W = \text{Span}\{u_1, u_2\}$

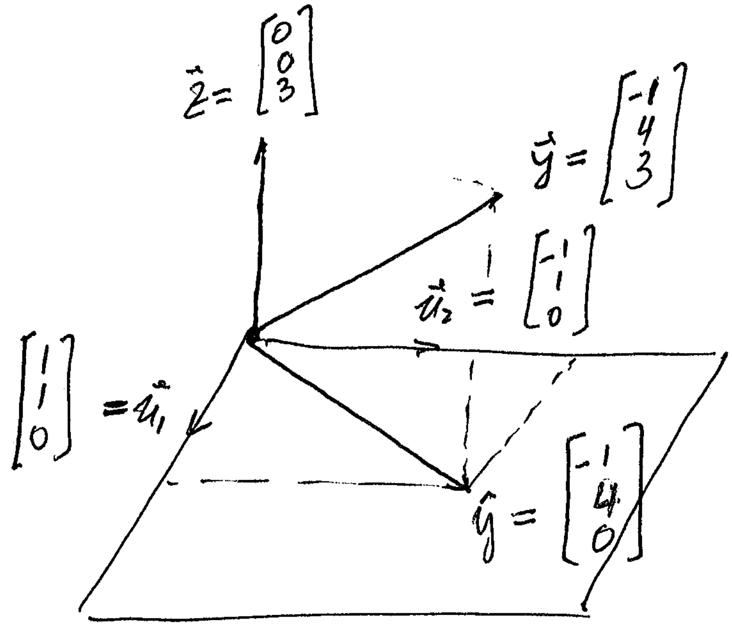
and

$$\text{proj}_W \vec{y} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

We also found that $\vec{y}$ can be decomposed as

$$\vec{y} = \underbrace{\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}}_{\in W} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}}_{\in W^\perp}$$





$$\vec{z} = \vec{y} - \hat{y}$$
$$= \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

i) The set $\{\vec{u}_1, \vec{u}_2, \vec{y}\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} \right\}$

form a basis for $\mathbb{R}^3$. (Prove it!). This basis is not orthogonal why?

ii) Construct an orthogonal basis for $\mathbb{R}^3$ from the vectors $u_1, u_2, \vec{y}$.

Answ:

i) $$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, $\{\vec{u}_1, \vec{u}_2, \vec{y}\}$ is linearly independent.

ii) we already saw $\vec{u}_1 \cdot \vec{u}_2 = 0$ they are orthogonal

However, $\vec{u}_1 \cdot \vec{y} = -1 + 4 = 3 \neq 0$

$\vec{u}_2 \cdot \vec{y} = 1 + 4 = 5 \neq 0$

Therefore, the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} \right\}$$ is not orthogonal.

Now, from our previous computation we found that

$$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \in W^{\perp} = (\overbrace{\text{Span}\{\vec{u}_1, \vec{u}_2\}}^{W})^{\perp}.$$

Therefore, $\vec{z} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$ is orthogonal to $u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and

$$\vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and the set

$$B_0 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$$ form an

orthogonal basis for $\mathbb{R}^3$.

Notice that $\vec{z}$ was obtained from $\vec{u}_1, \vec{u}_2$ and $\vec{y}$ from the equation

$$\vec{z} = \vec{y} - \hat{y} = \vec{y} - \text{proj}_W \vec{y}$$

or

$$\boxed{\vec{z} = \vec{y} - \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 - \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2}$$

This suggest a procedure to obtain orthogonal bases $B_0 = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ from not orthogonal basis $B = \{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$

<u>Example</u>

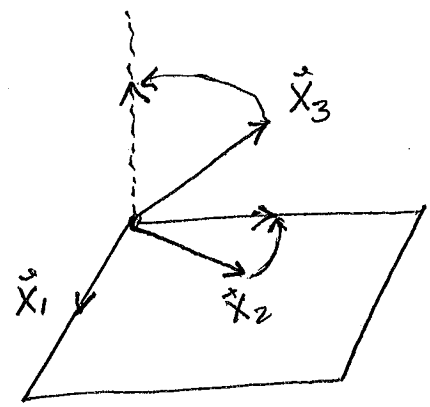
$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$$

i) Verify $B = \{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is not an orthogonal basis.

ii) Construct an orthogonal basis $B_0 = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ from $B = \{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$.

<u>answ:</u> i) $\vec{x}_1 \cdot \vec{x}_2 \neq 0, \quad \vec{x}_2 \cdot \vec{x}_3 \neq 0, \quad \vec{x}_1 \cdot \vec{x}_3 \neq 0$

ii)



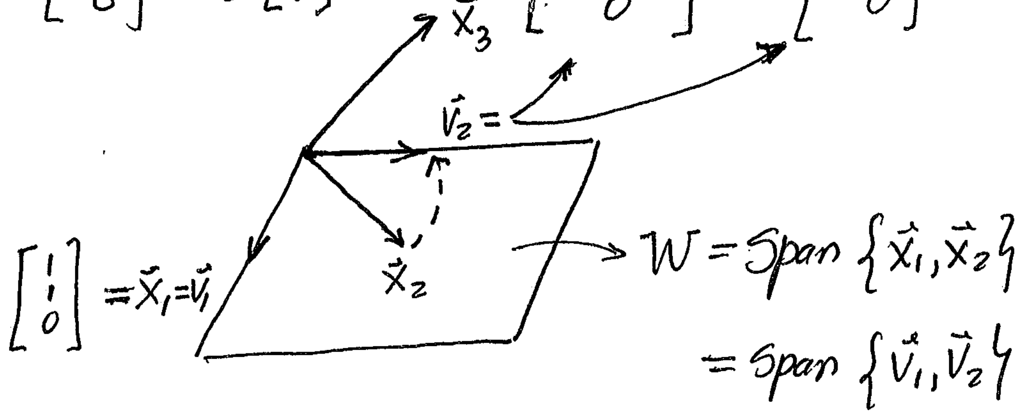
1st STEP: $\vec{v}_1 = \vec{x}_1$

2nd STEP: $\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{x}_1 = \vec{v}_1} \vec{x}_2 = \vec{x}_2 - \dfrac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$

or

$$\vec{v}_2 = \begin{bmatrix}-2\\3\\0\end{bmatrix} - \frac{[-2,3,0]\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}}{[1\,1\,0]\begin{bmatrix}1\\1\\0\end{bmatrix}}\begin{bmatrix}1\\1\\0\end{bmatrix} =$$

$$= \begin{bmatrix}-2\\3\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}-5/2\\5/2\\0\end{bmatrix} \sim \begin{bmatrix}-5\\5\\0\end{bmatrix}$$



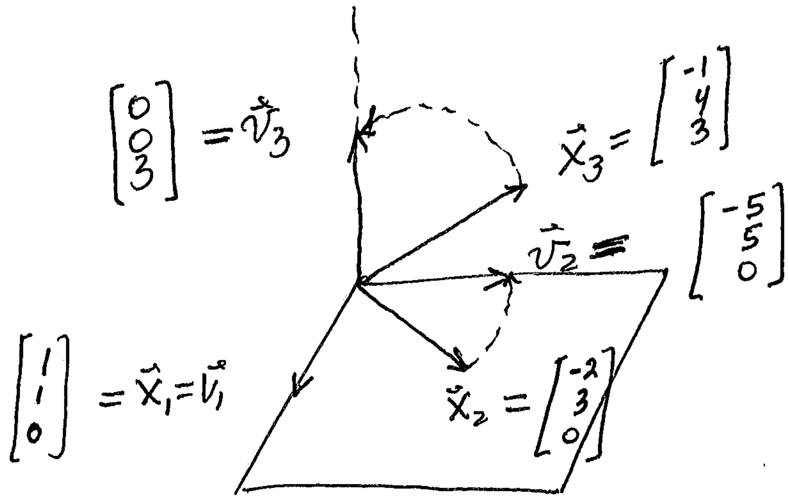
<u>STEP 3:</u> $\vec{v}_3 = \vec{x}_3 - \text{proj}_W\, \vec{x}_3$

$$= \vec{x}_3 - \frac{\vec{x}_3\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 - \frac{\vec{x}_3\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2$$

$$= \begin{bmatrix}-1\\4\\3\end{bmatrix} - \frac{[-1\,4\,3]\begin{bmatrix}1\\1\\0\end{bmatrix}}{[1\,1\,0]\begin{bmatrix}1\\1\\0\end{bmatrix}}\begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{[-1\,4\,3]\begin{bmatrix}-5\\5\\0\end{bmatrix}}{[-5\,5\,0]\begin{bmatrix}-5\\5\\0\end{bmatrix}}\begin{bmatrix}-5\\5\\0\end{bmatrix}$$

$$= \begin{bmatrix}-1\\4\\3\end{bmatrix} - \frac{3}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{\cancel{25}^{\,1}}{\cancel{50}_{\,2}}\begin{bmatrix}-5\\5\\0\end{bmatrix} = \begin{bmatrix}-1\\4\\3\end{bmatrix} - \begin{bmatrix}3/2\\3/2\\0\end{bmatrix} - \begin{bmatrix}-5/2\\5/2\\0\end{bmatrix}$$

or $\vec{v}_3 = \begin{bmatrix}0\\0\\3\end{bmatrix}$



Then, New $B_0 = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-5\\5\\0\end{bmatrix}, \begin{bmatrix}0\\0\\3\end{bmatrix} \right\}$

Notice

$$\text{Span}\left\{ \overset{\vec{x}_1}{\begin{bmatrix}1\\1\\0\end{bmatrix}}, \overset{\vec{x}_2}{\begin{bmatrix}-2\\3\\0\end{bmatrix}}, \overset{\vec{x}_3}{\begin{bmatrix}-1\\4\\3\end{bmatrix}} \right\} = \text{Span}\left\{ \overset{\vec{v}_1}{\begin{bmatrix}1\\1\\0\end{bmatrix}}, \overset{\vec{v}_2}{\begin{bmatrix}-5\\5\\0\end{bmatrix}}, \overset{\vec{v}_3}{\begin{bmatrix}0\\0\\3\end{bmatrix}} \right\}$$

The process applied is known as <u>Grand-Schmidt process</u> and it is Summarized in the following theorem:

<u>Thm 11</u> A Given basis $B = \{\vec{x}_1, .., \vec{x}_p\}$ of a Subspace $W \subset \mathbb{R}^n$ $(W \neq \{\vec{0}\})$ Can be transformed into an orthogonal basis for W $B_0 = \{\vec{v}_1, \vec{v}_2, .., \vec{v}_p\}$ by performing the following Computations:

$$\vec{v}_1 = \vec{x}_1, \quad \vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}, \quad \vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}$$

$$\vdots$$

$$\vec{v}_p = \vec{x}_p - \frac{\vec{x}_p \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} - \frac{\vec{x}_p \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} - \cdots - \frac{\vec{x}_p \cdot \vec{v}_{p-1}}{\vec{v}_{p-1} \cdot \vec{v}_{p-1}}$$

In addition, $\text{Span}\{\vec{x}_1, .., \vec{x}_k\} = \text{Span}\{\vec{v}_1, \vec{v}_2, .., \vec{v}_k\}$ for $1 \le k \le p$.

Section 6.4

Ex. 12 Find an orthogonal basis for the column space of the matrix

$$A = \begin{array}{c} \begin{matrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{matrix} \\ \begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix} \end{array}$$

First, we check that the set of columns of A is linearly independent

Row-reducing A

$$A \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & 0 & 6 \\ 0 & 2 & 3 \\ 0 & 2 & -3 \\ 0 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & -3 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & -3 \\ 0 & 0 & 6 \\ 0 & 0 & 6 \\ 0 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & -3 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So the original vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$ are linearly independent.

Now, we apply the G-S process:

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1} \vec{x}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

or $$\vec{v}_2 = \begin{bmatrix} 3 \\ -3 \\ 2 \\ 5 \\ 5 \end{bmatrix} - \overset{=4}{\frac{16}{4}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad W = \text{Span}\{\vec{v}_1, \vec{v}_2\} = \text{span}\{\vec{x}_1, \vec{x}_2\}$$

$$\vec{v}_3 = \vec{x}_3 - \text{proj}_W \vec{x}_3$$

or

$$\vec{v}_3 = \vec{x}_3 - \text{proj}_{\vec{v}_1} \vec{x}_3 - \text{proj}_{\vec{v}_2} \vec{x}_3 =$$

$$= \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{v_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$= \begin{bmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{bmatrix} - \frac{\overset{7}{\cancel{14}}}{\underset{2}{\cancel{4}}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{\overset{3}{\cancel{12}}}{\underset{2}{\cancel{8}}} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{bmatrix} + \begin{bmatrix} -7/2 \\ +7/2 \\ 0 \\ -7/2 \\ -7/2 \end{bmatrix} + \begin{bmatrix} 3/2 \\ -3/2 \\ -3 \\ -3/2 \\ -3/2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \\ -3 \\ 3 \end{bmatrix}$$

<u>Ex. 16</u> Find a QR factorization of the matrix A of Ex. 12.

We need to normalize the column vectors of matrix

$$B = \begin{array}{c} \begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array} \\ \begin{bmatrix} 1 & -1 & 3 \\ -1 & 1 & 3 \\ 0 & 2 & 0 \\ 1 & 1 & -3 \\ 1 & 1 & 3 \end{bmatrix} \end{array}$$

to obtain Q.

Notice that $\|\vec{v}_1\| = 2$, $\|\vec{v}_2\| = 2\sqrt{2}$, $\|\vec{v}_3\| = 6$

Then

$$Q = \begin{bmatrix} 1/2 & -1/2\sqrt{2} & 1/2 \\ -1/2 & 1/2\sqrt{2} & 1/2 \\ 0 & 1/\sqrt{2} & 0 \\ 1/2 & 1/2\sqrt{2} & -1/2 \\ 1/2 & 1/2\sqrt{2} & 1/2 \end{bmatrix}$$

We want obtain R Such that

$$A = QR$$

The columns of Q form an orthonormal basis for Col A. Then,

$$Q^T Q = I, \qquad \text{why?}$$

Therefore, to find R Such that $A = QR$ is enough to multiply both sides by Q^T as follows:

$$Q^T A = (Q^T Q) R = I R = R.$$

In our case,

$$R = \underbrace{\overbrace{\begin{bmatrix} 1/2 & -1/2 & 0 & 1/2 & 1/2 \\ -1/2\sqrt{2} & \frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 1/2 & 1/2 & 0 & -1/2 & 1/2 \end{bmatrix}}^{Q^T}}_{3\times 5} \underbrace{\begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix}}_{5\times 3} = \begin{bmatrix} 2 & 8 & 7 \\ 0 & 2\sqrt{2} & 3\sqrt{2} \\ 0 & 0 & 6 \end{bmatrix}$$

Upper triangular matrix
Invertible.

The factorization described in the previous exercise is of practical Value for solving equations and finding eigenvalues. It's called "the QR Factorization".

Thm 12 $A_{m\times n}$ has n linearly independent columns.
Then

$$A = QR$$

Where $Q_{m\times n}$ matrix: Whose columns form an orthonormal basis for $Col(A)$

and $R_{n\times n}$ matrix: upper Triangular and invertible.
With positive diagonal entries.